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THE PENTA-GEKSAEDR

Now among in the physicist, crystallographist and to the mathematician it is known [1] seven crystallography systems: *triclinic, monocline, ortorombic, tetragonal, cubic, trigonal and hexagonal.*

1.In geometry [2] there are only five correct polyhedrons at which all edge are equal correct poligons and all many-sided corners are equal.

Correct polygons (Platon's Body) are known very much for a long time. They have been drawn by it in IV century B.C. These are five geometrical objects, represented on fig.1. Here a tetrahedron, a cube, an octahedron, the pentagon-dodecahedron and the ikosaedron.

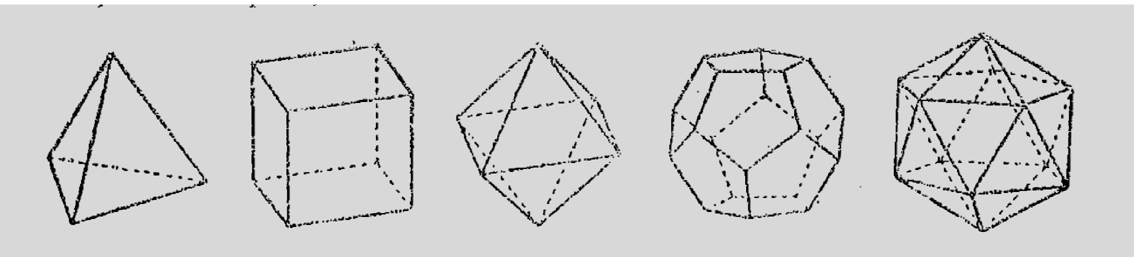


Fig1. Correct polyhedrons: 1--tetrahedron, *2*-cube, 3-octahedron, 4-dodecahedron, 5-ikosaedr
(Figures are represented in different scales)

On the astral value these bodies were identified with different elements: the fire, the earth, the air, the Universe and the water. On fig.1 they are represented in different scales. However, if for the basic unit of measure to choose length of an edge of figures (as it usually and becomes) and to represent all of them in identical linear scale they will look absolutely variously. The huge crystal of the pentagon-dodecahedron was identified with all Universe.

Elements of symmetry of correct polyhedrons are presented in table 1.

For pure polyhedrons Euler's theorem is known: if e - number of peaks of a polyhedron, f - number of sides and k-number of edges,

$$e - k + f = 2 \tag{1}$$

The theorem of **<u>the author of the present work</u>** (till now yet not published) is: if at the same designation, n - number of edge at peaks, that

$$(e \cdot n) : k \ =2 \qquad (2)$$

The theorem can be proved the direct account.

Elements of pure polyhedrons are presented in table 1

Elements of pure polyhedrons (*a* - length of an edge) Table 1

Name	Number of sides f and their form	Number of edges, k	Number of peak, e	Complete surface	Volume
~~1~~	~~2~~	~~3~~	~~4~~	~~5~~	~~6~~
Tetrahedron	4 triangles	6	4	1,7321 a^2	0,1179 a^3
Cube	6 quadrates	2	8	6 a^2	a^3
Octahedron	8 triangles	12	6	3,4641 a^2	0,4714 a^3
Dodecahedron	12 pentagons	30	20	20,6457 a^2	7,6631 a^3
Icosahedrons	20 triangles	30	12	8,6603 a^2	2,1817 a^3

2. Consideration of features of the pentagonal symmetry, made in [3] has shown, that in pentagonal space it is possible to construct set of pentagons on a plane and to fill with them plane XY without intervals from - ∞ to + ∞ on both axes. Some feature consists that plane filling occurs special triangles pentagonal symmetry, the primary pentagon is made of 5 which pieces. Such triangle in [3] is named *<u>trefs</u>* (a triangle made of atoms Ferrum (iron) and Sulphur FeS_2)

3. "Historically the conclusion about impossibility of axes of the fifth order and more the sixth has been deduced from the law of rational indexes"[1].

However, well-known [2], that in geometry there are five correct polyhedrons at which all sides equal correct polygons and all many-sided corners are equal. They are represented on fig.1. Among these five polyhedrons is available two correct polyhedrons (4 and 5) under names "pentagon-dodecahedron (dodecahedron)" and "the ikosaedron", greatest of all figures. They has SIX rotary axes of the FIFTH order(!), 10 axes of 3 and 15 axes of the second order and also 30 planes.

Besides, in geology, the mineralogy and metallurgy [4], the mineral Pirit is well known.

In a geological museum I also have seen a brilliant mineral Pirit FeS_2 *(pўr-* fire*)* A synonym - *iron firestone.* Shape the pentagon-dodecahedron, doubles on (110). hardness 6,65, it is fragile. The colour is gold–yellow. (Photo of Pirit on fig.5).

Pirit is the typical representative of a crystal of pentagonal symmetry. He crystallises in the form of a pentagon-dodecahedron**.** The most widespread sulphide. It is formed in the diversified geological conditions: magmatic, metamorphic, hydrothermal, exogenetic.

. Therefore the conclusion from the law of rational numbers on impossibility of axes of symmetry 5 order is quite erroneous.

From the previous description it is obvious, that it is impossible to fill a plane with correct pentagons without backlashes between them. However, it is known, that a plane it is possible to fill with the correct hexagons, pressed one to another without backlashes

3. The icosahedron it is possible to enter in a dodecahedron then icosahedron tops will be combined with the centres of sides of a dodecahedron.

Thus in the Pentagon-dodecahedron between a icosahedrons surface and a Pentagon-dodecahedron surface the free space (a difference of volumes) is formed as though. However, this space appears made of 60 tetrahedrons, edges in which are equal to length of radius of the circle described round a pentagon of a side of the Pentagon-dodecahedron and 60 special tetrahedrons in which two opposite edges are equal to the radiuses of the described circle specified above, and two other edges are equal to length of edges of the Pentagon-dodecahedron. By means of system of these tetrahedrons there is a combination of longer edges of the Pentagon of a dodecahedron with smaller on length (in absolute units) edges of a icosahedrons.

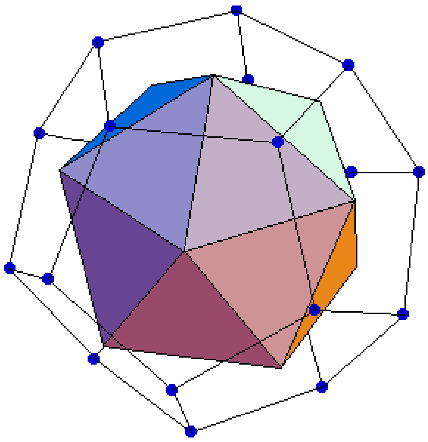
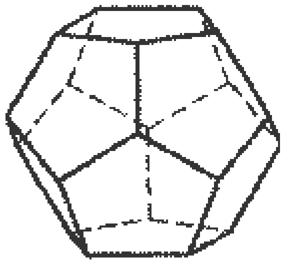


Fig.2.A icosahedron in the Pentagon-dodecahedron

The tetrahedrons of one party in 60 tetrahedrons alternate with 60th tetrahedrons of the second party, settling down inside on 12 sides of the Pentagon-dodecahedron.

On fig. 2 the kind of the Pentagon of a dodecahedron and its basic characteristic sizes is resulted.



$$S = 3a^2\sqrt{5(5+2\sqrt{5})} \quad (1)$$

$$V = \frac{a^3(15+7\sqrt{5})}{4} \quad (2)$$

$$R = \frac{a\sqrt{3}(1+\sqrt{5})}{4} \quad (3)$$

$$r = \frac{a\sqrt{10(25+11\sqrt{5})}}{20} \quad (4)$$

Fig. 3. The Pentagon a dodecahedron and its characteristic sizes: *S* - the surface area, figure *V*-volume, *R* - radius of the described sphere, *r* - radius of the entered sphere, *a* - this is size of an edge of a figure.

4. How be described in job [2] pentagonal side of the Pentagon of a dodecahedron can it is made of five "wrong" triangles - two parties at each triangle are equal to radius of the circle described round a pentagon *b*, and the third is equal $2b \cdot \sin 36^o$.

Fig.4. A druse from crystals Pirit

On fig. 3 it is represented a druse from three accrete crystals Pirit (two big and one small - below). These are the most beautiful crystals of bright golden colour with excellent metal shine of sides. This golden colour and magnificent shine are saved on air at a mineral long time (some tens years) without tarnishing.

5. The icosahedrons and the Pentagon-dodecahedron have formulas of elements of symmetry:

The icosahedrons	$6g_5 + 10g_3 + 15g_2 + 30P + C$	(5)
The Pentagon-dodecahedron	$6g_5 + 10g_3 + 15g_2 + 30P + C$	(6)

As we see, symmetry formulas, at both geometrical bodies, are absolutely identical. Distinction consists only that axes of symmetry of the fifth order at icosahedron pass through opposite tops, and at the Pentagon-dodecahedron - through the opposite centres of sides.

6. I have set the task to construct the volume body having both pentagonal and **geksagonal** sides. Such body, of course, is not entered in Platon's bodies. But, probably, it can have application in a life of a modern society. Similar, also, that this problem did not dare yet in science and education system till now.

And preconditions for the successful decision consist in the following: on a plane it is possible to construct a circle (fig. 5) with radius *b* and to enter in it a correct pentagon with edges *a*. To all edges of the pentagon to attach five hexagons with the parties equal *a*.

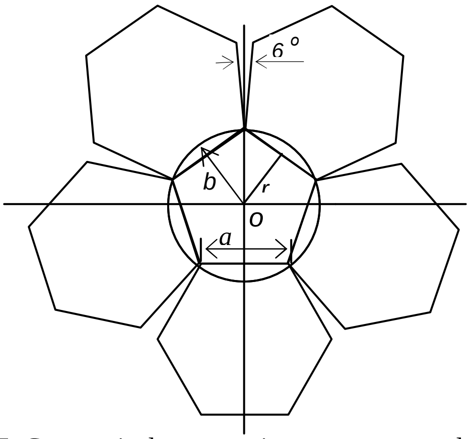


Fig. 5. Geometrical construction a pentagon and five hexagons with the parties *a* .

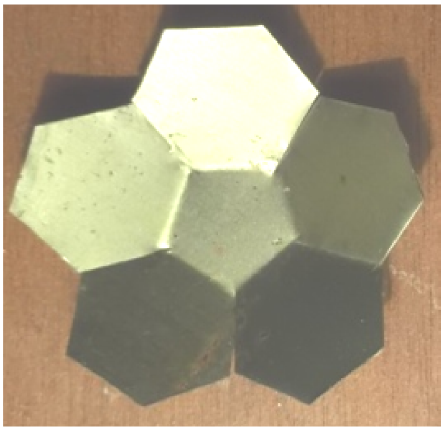


Fig. 6. The model of the socket which is turning out from a figure, drawn on fig.5 after turn of all hexagons petals to clamping.

As we see between 6 coal petals backlashes in 6^o are formed.

If all of them to turn to clamping edges the figure similar to model will turn out, represented on fig. 6.

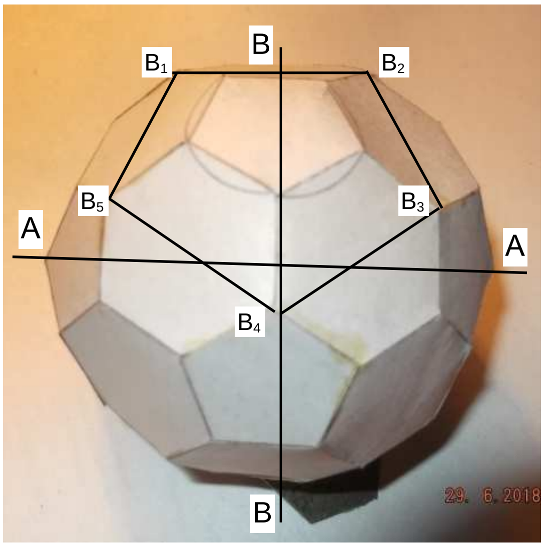


Fig. 7. A penta-hexahedron with the image of lines of cut A-A and sections B_1 - B_5

7. The image on fig. 5 and 6 is one of the major elements of the volume body represented at the left. This body it is possible to name the PENTA-GEXAEDR.

Its image is shown on fig. 7. Here it is possible to see pentagon sides of a body, round each of which is available five hexagon sides. Thus round each pentagon sides the socket is formed of hexagons, as on fig. 6

Such body has 12 pentagonal sides and 20 hexagonal sides. Thus, total of sides of the Penta-hexahedron 32 pieces.

Plane B_1-B_5 which is passing through diagonals hexagonal sides in space it is parallel to the top pentagonal side of the Penta-hexahedron and both of them limit the spatial figure named in the mathematician [5] as «the truncated pyramid» (fig.113).

To symmetrically it below on an axis of 5th order there is same «a truncated pyramid». Besides, round each of 12 pentagonal sides, can be constructed, same «the truncated pyramid». That is, in volume Penta-Geksaedra can be constructed such 12 of "the truncated pyramids".

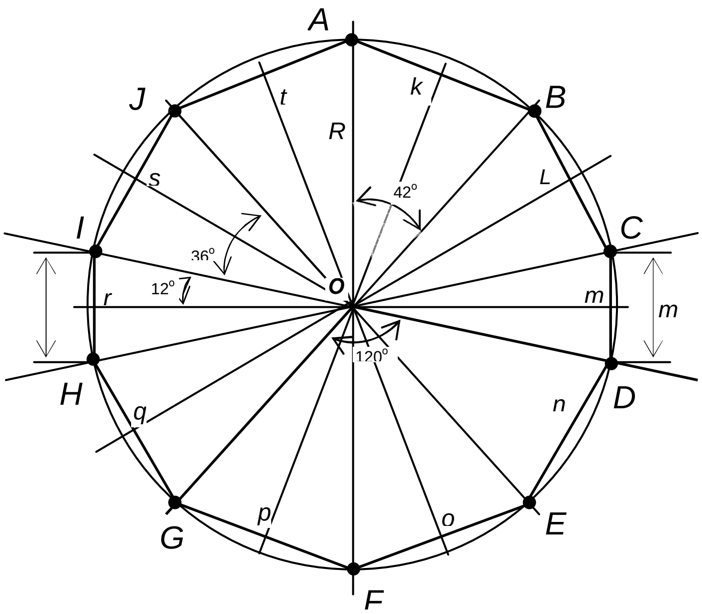


Fig. 8. A cut of the Penta-hexahedron in the area of A-A.

8. It is possible to execute a cut of a body fig. 7 in the area of A-A, vertically screen planes. Figure *ABCDEFGHIJ* (fig. 8) thus turns out. On fig. 7 cut A-A passes through the middle of its edges hexagon sides, and the radius of the described circle of this section is *R* (fig. 8).

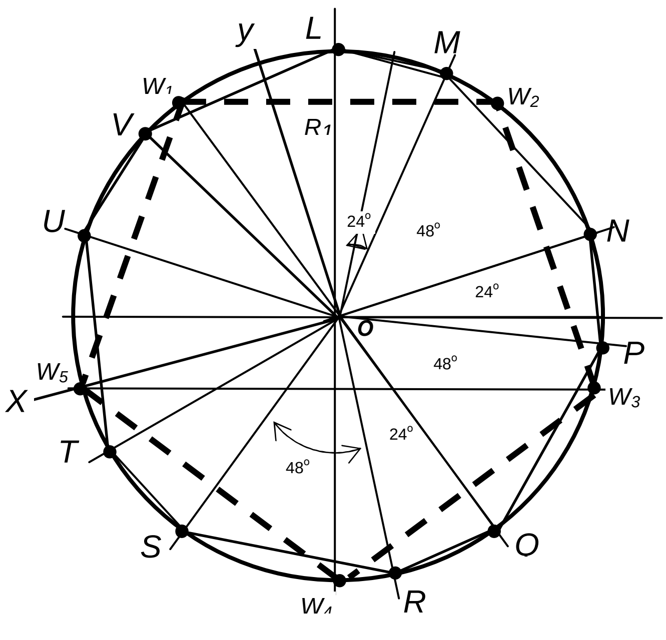


Fig. 8. A cut of the Penta-hexahedron in the area of B-.

Here there are four pieces *AB, AJ, FG* and *FE,* equal everyone to two apothems of hexagons of the Penta-hexahedron.

Besides there are four pieces *BC,JI, DE* and *HG* , equal .

$$BC = JI=DE=HG=b+b\cdot Sin54^{o} \qquad (7)$$

And on a horizontal two pieces *CD* and *HI* , equal

$$CD=HI=m=a=2bSin36=2R\cdot Sin12^{o} \qquad (8)$$

Therefore radiuses of the described circle of section on fig. 9

$$OL=OM=ON=OP=OQ=OR=OS=OT=OU=OV=R_1 .$$

From fig. 8 it is had

$$\angle AOB=\angle AOJ=\angle GOF=\angle FOE= 42^o \qquad (9)$$

$$\angle JOI=\angle BOC=\angle HOG=DOE= 36^o \qquad (10)$$

$$\angle COD=\angle HOI=24^o \qquad (11)$$

$$\angle DOG=\angle BOI=120^o \qquad (12)$$

and on fig. 9 it is had

$$LM=NP=QR=ST=UV=a \qquad (13)$$

$$MN=PQ=RS=TU=VL=2a \qquad (14)$$

$$\angle LOM=\angle NOP=\angle QOR=\angle SOT=\angle UOV=24^o \qquad (15)$$

$$\angle MON=\angle POQ=\angle ROS=\angle TOU=\angle VOL=48^o \qquad (16)$$

$$\angle W_1W_2W_3=\angle W_2W_3W_4=\angle W_3W_4W_5=\angle W_4W_5W_1=\angle W_5W_1W_2=108^o \qquad (17)$$

Such cuts which is represented on fig. 8, in the Penta-hexahedron it is possible to make on 5 pieces about each pentagon.

In total pentagons in the Penta-hexahedron there are 12 pieces. And four pentagonal sides are involved in each cut of type fig. 8. Thus, total of independent cuts as on fig.8 in Penta-Geksaedr it is possible to construct N pieces

$$N=12 \times 5 : 4 =15 \qquad (18)$$

Such cut as on fig. 8 defines a plane of symmetry of R. Therefore in the formula of elements of symmetry the Penta-hexahedron is had 15P.

In total hexahedron sides in the Penta-hexahedron are 20 pieces. Through the centre of two opposite sides it is possible to spend a rotary axis of 3rd order. Therefore in the formula of elements of symmetry the Penta-hexahedron it is had $10g_3$.

Through the centre of joints of two next 6 coal sides it is possible to spend a rotary axis of 2nd order, and there are 5 places where 6 squares are involved twice. Therefore in the formula of elements of symmetry of the Penta-hexahedron it is had $15g_2$.

9. The external surface of Penta-hexahedron S_{outer} consists from 12 pentagons and 20 hexagons

$$S_{outer} = 12\, S_5 + 20\, S_6 \qquad (19)$$

where S_5 - the area of pentagon

$$S_5 =1/2 \cdot 5a\, r_5 =5/4\, a^2 Sin\, 54^o /Sin 36^o \qquad (20)$$

and S_6 - the area of hexagon

$$S_6 =3\, a\, r_6 =3a^2 Sin\, 60^o \qquad (21)$$

where r_5 - a pentagon apothem, and r_6 - a hexagon apothem, that is, the external surface of the Penta-hexahedron is counted up under the formula

$$S_{outer} = 15\ a^2 Sin\ 54^o / Sin36^o + 60\ a^2\ Sin\ 60^o = 69{,}595\ a^2 \qquad (22)$$

10. From the drawing fig. 8 it is possible to count up Penta-hexahedron volume on the pyramids leaving the centre about fig. 8.

The volume of a pyramid V on [5] is equal

$$V = 1/3\ S \cdot h \qquad (23)$$

where S - the area of the basis of a pyramid, and h pyramid-height.

Thus, we will count up pentagon pyramid volumes V_5 and hexagon pyramids volumes V_6 under the formula (18) with use of sizes from fig. 5 and fig

$$\text{Edge}\ a = 2b \cdot Sin36^o, \qquad (24)$$

and from fig. 8

$$a \cdot 1/2 \quad = R \cdot Sin\ 12^o \qquad (25)$$

Then heights of pyramids h_6 и h_5

$$h_6 = \frac{a\,Sin69^0}{2Sin12^0} \qquad (26)$$

$$h_5 = \frac{a\,Sin72^0}{2Sin12^0} \qquad (27)$$

Knowing the areas of the bases and height of pyramids, we will find volumes of single pyramids under the formula (23)

$$V_6 = \frac{1}{2} a^3\ \frac{Sin60^o\,Sin69^o}{Sin12^o} = 56{,}044\ \ a^3 \qquad (28)$$

$$V_5 = \frac{5}{24} a^3\ \frac{Sin54^0\,Sin72^0}{Sin36^0\,Sin12^0} \qquad (29)$$

Total amount of all 6 and 5 coal pyramids in the sizes of an edge and the Penta-hexahedron we will find under the formula

$$V_{total} = \sum_{l=1}^{12} \frac{5}{24} a^3\,\frac{Sin54^o\,Sin72^o}{Sin36^o\ Sin12^0} \quad + \sum_{k=1}^{20} \frac{1}{2} a^3\,\frac{Sin60^o\,Sin69^o}{Sin12^o} = 56{,}044\ \ a^3 \qquad (30)$$

And the Penta-hexahedron total amount in the sizes of radius R looks like

$$V_{total} = \sum_{l=1}^{12} \frac{5}{3} R^3\,\frac{Sin54^o\,Sun^2 69\ Sin^2 12^o}{Sin36^o} + \sum_{k=1}^{20} 4\,R^3 Sin^2 12^0\ Sin\ 69^0 \qquad (31)$$

12. And It is necessary to compare also total amount of the Penta-hexahedron to volume of sphere [5] with radius equal R

$$V_{sphere} = 4/3\ \pi\ R^3 = 4{,}183\ R^3 \qquad (32)$$

with volume of the Penta-hexahedron which under the formula (31) has size

$$V_{total} = 4{,}2644\ R^3$$

The volume of the Penta-hexahedron (31) exceeds volume of sphere (32) on size 0,0754 R^3, that makes 7,54 %. There is it because some of the sizes of parts of the Penta-hexahedron out of a cut on fig. 8 have great values of radiuses (on ledges of tops), in comparison with radius of this described circle. It is necessary to notice, that Platon's all bodies have spheres describing them more than figures described by them spheres.

13. On fig. 8 there are two triangles *HOI* and *COD* c area $S = Om \cdot mC$ everyone. However, these areas do not create any volumes in the Penta-hexahedron as their area on the Penta-hexahedron surface is equal to zero. Such zero volumes in the Penta-hexahedron 30 pieces are.

14. It is important to understand existence of corners 120^o on fig. 8. These corners, leave the centre of a hexagon and pass through the centre of three pentagons surrounding hexagons. Thus, the line connecting the centre of each hexagon with the centre About and the centre of a symmetric hexagon on an opposite side the Penta-hexahedron, is a rotary axis of symmetry of the third order g_3. Therefore round each hexagon 3 pentagons are visible. Quantity of such axes in the Penta-hexahedron (see (13)) equally

$$N_1 : 2 = 20 : 2 = 10 \qquad (33)$$

They form quantity of rotary axes of the third order, similarly (5) and (6), as $10g_3$.

Through the centre of each pentagonal side in the Penta-hexahedron there passes a rotary axis of the fifth order g_5. Considering, that pentagons in Penta-Geksa-edre 12, we receive quantity of rotary axes of the fifth order $6g_5$.

Thus, the formula of elements of symmetry of the Penta-hexahedron looks like (34) (for comparison symmetry icosahedrons parametres a are more low resulted) (5)

Penta-hexahedron **$6g_5 + 30g_3 + 5g_2 + 6P + C$** (34)

Icosahedron $6g_5 + 10g_3 + 15g_2 + 30P + C$ (35)

In the Penta-hexahedron the identical quantity of elements of symmetry of 5th order in comparison with Icosahedron, three times is more than elements of symmetry of 3rd order, but in 3 times there are less than elements of symmetry of 2nd order and in 5 times of less planes of symmetry.

Continuing table 1 from [2], we will enter in it one line with Penta-hexahedron parametres, three times is more than elements of symmetry of 3rd order, but in 3 times there are less than elements of symmetry of 2nd order and in 5 times of less planes of symmetry.

Elements of pure polyhedrons (*a* - length of an edge)

Table 2

Name	Number of sides *f* and their form	Number		Complete surface	Volume
		of edges, *k*	of peak, *e*		
Tetrahedron	4 triangles	6	4	1,7321 a^2	0,1179 a^3
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Octahedron	8 triangles	12	6	3,4641 a^2	0,4714 a^3
Dodecahedron	12 pentagons	30	20	20,6457 a^2	7,6631 a^3
Icosahedrons	20 triangles	30	12	8,6603 a^2	2,1817 a^3
Penta-geksedron	32=12 pentagons and 20 geksagons	90	60	69,595 a^2	56,044 a^3

Thus, we have at the Penta-hexahedron of more sides, than at greatest of Platon's bodies on 12 sides, it is more than edges and tops in 3 times, more the total surface in 3,37 times and has more than total Volume in 7,313 times.

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